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On the kernels of the incidence matrices of graphs

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Abstract

This paper proposes and investigates a problem about the sizes of the entries of the vectors in the kernel of the incidence matrix of an arbitrary graph.

1. Introduction

The purpose of the present article is to propose and investigate a certain problem about the structure of the kernel of the incidence matrix of any undirected graph. Roughly speaking, we are interested in the minimum of the maximum weight of vectors in the basis of the kernel for some weights. (See Section 2 for a more precise formulation.) This seemingly innocent problem has emerged through our study of Hodge cycles on certain abelian varieties of CM-type in [5], and turns out to be quite efficient for the investigation of the combinatorial aspect of the structure of the ring of Hodge cycles. In this article, we consider the problem from a graph theoretical viewpoint, so that any reader with an elementary knowledge about the theory can appreciate it.

The plan of the paper is as follows. In Section 2, we introduce the notion of *h-degeneracy* and *w-dominatedness*, and formulate the problem of our main concern in terms of this notion. In Section 3, we prove that every graph is one-degenerate. This result has an important application to the theory of Hodge cycles on abelian varieties (see Remark 2.2). Namely, if an abelian variety A of CM-type is associated to a graph as in [5], then every Hodge cycle on A^n, $n \geqslant 1$, is already realized on A itself. In Section 4 we show that the complete graph K_m and the complete bipartite graph $K_{m,m}$

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are two-dominated for any $m \geqslant 3$. This result has also an application to the theory of Hodge cycles on certain abelian varieties.

2. Problem setting

In this section, we formulate the main problem of the paper and illustrate it through some examples.

Let $G=(V,E)$ be an undirected graph with $V=\{v_i; 1 \leqslant i \leqslant m\}$, $E=\{e_j; 1 \leqslant j \leqslant n\}$. Let B be the incidence matrix of G whose entries b_{ij} are given by

$$b_{ij}=\begin{cases} 1 & \text{if } v_i \in e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{V}(G)$ denote the free $\mathbf{Z}$-module consisting of formal $\mathbf{Z}$-linear combinations $\sum_{1 \leqslant i \leqslant m} a_i v_i$, $a_i \in \mathbf{Z}$, and $\mathbf{E}(G)$ the free $\mathbf{Z}$-module of formal $\mathbf{Z}$-linear combinations $\sum_{1 \leqslant j \leqslant n} b_j e_j$, $b_j \in \mathbf{Z}$. For any pair $(v,e) \in V \times E$, let

$$[v,e]=\begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{\partial}: \mathbf{E}(G) \to \mathbf{V}(G)$ denote the $\mathbf{Z}$-linear map defined by $\bar{\partial}(e)=\sum_{v \in V} [v,e]v \in \mathbf{V}(G)$. Therefore, the incidence matrix B is regarded as the representation matrix of the linear map $\bar{\partial}$ with respect to the natural bases. For any $\mathbf{e}=\sum_{e \in E} a_e e \in \mathbf{E}(G)$, let $H(\mathbf{e})=\max\{|a_e|; e \in E\}$, and call it the *height* of the element $\mathbf{e}$. Furthermore, let $W(\mathbf{e})=(\sum_{e \in E} |a_e|)/2$, and call it the *weight* of the element $\mathbf{e}$. For any finite subset $S \subset \mathbf{E}$, let $H(S)=\max\{H(\mathbf{e}); \mathbf{e} \in S\}$, and $W(S)=\max\{W(\mathbf{e}); \mathbf{e} \in S\}$, and call them the *height* (resp. *weight*) of S. We are interested in the minimal height and weight of the spanning subsets of the kernel $\mathbf{K}(G)$ of $\bar{\partial}: \mathbf{E}(G) \to \mathbf{V}(G)$ and introduce the following notion:

Definition 2.1. When the minimum of $H(S)$ with S running through the class of finite spanning subsets of the kernel $\mathbf{K}(G)$ is equal to h, we say the graph G to be h-degenerate and write $h=h(G)$. When the minimum of $W(S)$ with S running through the class of finite spanning subsets of the kernel $\mathbf{K}(G)$ is equal to w, we say the graph G is w-dominated and write $w=w(G)$.

The problems we want to investigate are the following two related ones:

(1) To characterize the one-degenerate graphs.
(2) To characterize the two-dominated graphs among one-degenerate ones.

Remark 2.2. As is considered in [5], these problems have their origins in the study of Hodge cycles on abelian varieties of CM-type. When the automorphism group of a graph G satisfies a certain condition, we can associate to G an abelian variety A of CM-type, for which the structure of the ring of Hodge cycles $\mathscr{B}^\bullet(A^n)$, $n \geqslant 1$, is depicted through the $\mathbf{Z}$-module $\mathbf{K}(G)$. In this circumstance, if the graph is h-degenerate,

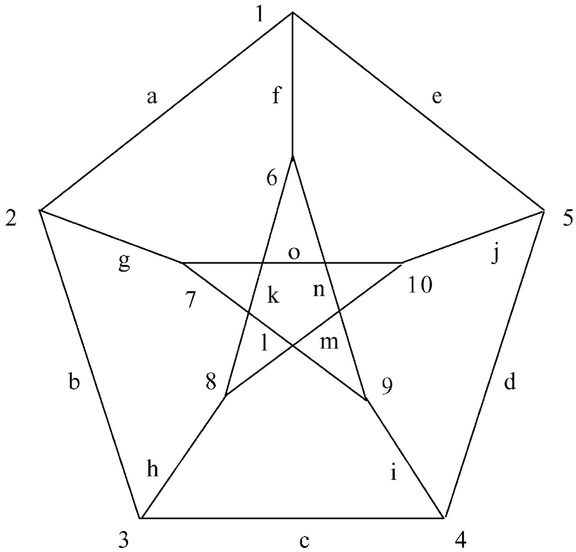

Fig. 1.

then every Hodge cycle on A^n, $n \geqslant 1$, is already realized in A^h, and if the graph is w-dominated, then the whole ring of Hodge cycles $\mathscr{B}^\bullet(A^n)$, $n \geqslant 1$, is generated by the Hodge cycles of codimension up to w.

Example 2.3. *The Petersen graph* O_3 (*Fig.*1):

The incidence matrix is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Here the columns (resp. rows) are indexed in alphabetical (resp. numerical) order. One can easily check that the kernel of B is spanned by five elements

$$a - b + h - m + j - e, \quad b - c + i - n + f - a, \quad c - d + j - o + g - b,$$
$$d - e + f - k + h - c, \quad e - a + g - l + i - d$$

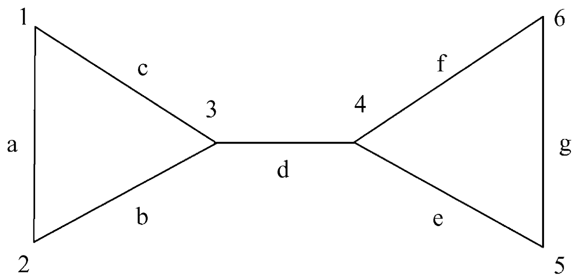

Fig. 2.

in $\mathbf{E}(O_3)$. Since we see that there is no element of weight 2 in the kernel, we obtain the following:

$$\text{the Petersen graph is one-degenerate and three-dominated.} \tag{2.1}$$

Example 2.4. Let $G_{\deg} = (V, E)$ be the following graph with $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{a, b, c, d, e, f, g\}$ $(Fig. 2)$:

The incidence matrix B is found to be

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

One can easily check that the kernel $\mathbf{K}(G_{\deg})$ is of rank one and spanned by $a - b - c + 2d - e - f + g \in \mathbf{E}(G_{\mathrm{def}})$. Therefore,

$$\text{the graph } G_{\deg} \text{ is two-degenerate and four-dominated.} \tag{2.2}$$

Remark 2.5. For any graph G consisting of two disjoint odd cycles joined by a path, one can easily check by a similar argument that the kernel $\mathbf{K}(G)$ is of rank one and spanned by an element of height two (see the proof of Theorem 3.4 below).

Remark 2.6. The $\mathbf{Z}$-linear map $\bar{\partial} : \mathbf{E}(G) \to \mathbf{V}(G)$ coincides with the usual boundary map when reduced modulo 2. Hence its kernel mod 2 consists of *cycles* in the usual sense, and is of course investigated extensively in the literature. Thus, our kernel may be regarded as giving us a bit more delicate invariant for graphs than the usual cycle space.

3. Degeneracy of graphs

The purpose of this section is to prove that every graph is two-degenerate and every two-connected graph is one-degenerate.

Let G be a graph on the vertex set $V=\{v_1,\ldots,v_m\}$ and $E=\{e_1,\ldots,e_n\}$ be the set of edges. For any $\mathbf{u}=\sum_{1\leqslant j\leqslant n} b_j e_j \in \mathbf{E}(G)$, let $\mathrm{supp}(\mathbf{u})=\{j; b_j\neq 0\}$ and call it the support of $\mathbf{u}$. Every vector $\mathbf{u}\in\mathbf{E}(G)$ can be written uniquely as $\mathbf{u}=\mathbf{u}^+-\mathbf{u}^-$ where $\mathbf{u}^+$ and $\mathbf{u}^-$ have positive coefficients and have disjoint support. A nonzero element $\mathbf{u}\in\mathbf{K}(G)$ is said to be an *elementary vector* if its support is minimal with respect to inclusion. An *elementary integral vector* of $\mathbf{K}(G)$ is an elementary vector with relatively prime coefficients. For any nonzero element $\mathbf{u}\in\mathbf{K}(G)$, let G_u denote the subgraph of G having vertex set

$$V_u=\{v\in V\colon\ v \text{ is incident with some edge } e_j \text{ with } j\in\mathrm{supp}(\mathbf{u})\}$$

and edge set

$$E_u=\{e_j\in E; j\in\mathrm{supp}(\mathbf{u})\}.$$

The following proposition gives a geometric description of the elementary vectors.

Proposition 3.1 (Villarreal [6, Proposition 4.2]). *An element $\mathbf{u}\in\mathbf{K}(G)$ is an elementary vector if and only if G_u is an even cycle or a connected graph consisting of two edge disjoint odd cycles joined by a path.*

As an immediate consequence, we have the following.

Proposition 3.2 (see Villarreal [6, Corollary 4.1]). *Every graph is two-degenerate.*

Proof. Any element $\mathbf{u}$ in a $\mathbf{Z}$-basis of $\mathbf{K}(G)$ is necessarily an elementary integral vector. Therefore, the assertion follows from Proposition 3.1 and Remark 2.3. □

Next we show that any two-connected graph is one-degenerate. A graph is said to be *two-connected* if it is connected, has at least three vertices and contains no cutvertex. The following characterization of two-connectivity is crucial.

Proposition 3.3 (Bollobas [1, III.2, Corollary 6]). *A graph is two-connected if and only if it has at least two vertices and any two vertices can be joined by two independent paths.*

(A pair of paths is said to be *independent* if each vertex belonging to both paths is an endvertex of both.) Now we can prove the following.

Theorem 3.4. *Every two-connected graph is one-degenerate.*

Proof. For any integers $a,b,c\geqslant 1$, let $G_{a,b,c}$ be the graph with

$$V(G_{a,b,c})=\{v_0,\ldots,v_{2a},w_0,\ldots,w_{2b},\ldots,x_1,\ldots,x_c\}.$$

$$E(G_{a,b,c}) = \{v_i v_{i+1}; 0 \leqslant i \leqslant 2a\} \cup \{w_j w_{j+1}; 0 \leqslant j \leqslant 2b\}$$
$$\cup \{v_0 x_1, x_1 x_2, \ldots, x_{c-1} x_c, x_c w_0\}.$$

(Here we employ the convention that $v_{2a+1} = v_0, w_{2b+1} = w_0$.) Therefore, $G_{a,b,c}$ consists of two disjoint odd cycles joined by a path. Let

$$\begin{aligned}\mathbf{u} = &\sum_{0 \leqslant i \leqslant 2a} (-1)^i (v_i v_{i+1}) \\ &+ 2\left\{-(v_0 x_1) + \sum_{1 \leqslant k \leqslant c-1} (-1)^{k-1} (x_k x_{k+1}) + (-1)^{c-1} (x_c w_0)\right\} \\ &+ (-1)^c \sum_{0 \leqslant j \leqslant 2b} (-1)^j (w_j w_{j+1}) \in \mathbf{E}(G_{a,b,c}).\end{aligned}$$

By Proposition 3.1, we may assume that our graph G contains $G_u = G_{a,b,c}$ as a subgraph for some a, b, c, and we are reduced to showing that $\mathbf{u} \in \mathbf{K}(G_{a,b,c})$ is expressed as a linear combination of elements $\in \mathbf{K}(G)$ of height one. It follows from Proposition 3.3 that there exists a path P from v_1 to w_1 independent of the path $\{v_1 v_0, v_0 x_1, x_1 x_2, \ldots, x_{c-1} x_c, x_c w_0, w_0 w_1\} \subset E(G_{a,b,c})$. We put

$$P = \{v_1 y_1, y_1 y_2, \ldots, y_{d-1} y_d, y_d w_1\}.$$

We treat the two cases (1) $c + d$ is even, (2) $c + d$ is odd, separately.

Case 1: $c + d$ is even: Let

$$\begin{aligned}\mathbf{u}_1 = &\left\{(v_1 v_0) - (v_0 x_1) + \sum_{1 \leqslant k \leqslant c-1} (-1)^{k-1} (x_k x_{k+1}) + (-1)^{c-1} x_c w_0 + (-1)^c w_0 w_1\right\} \\ &+ \left\{-(v_1 y_1) + \sum_{1 \leqslant l \leqslant d-1} (-1)^{\ell-1} (y_\ell y_{\ell+1}) + (-1)^{d-1} (y_d w_1)\right\},\end{aligned}$$

$$\begin{aligned}\mathbf{u}_2 = &\sum_{1 \leqslant i \leqslant 2a} (-1)^i (v_i v_{i+1}) \\ &+ \left\{-(v_0 x_1) + \sum_{1 \leqslant k \leqslant c-1} (-1)^{k-1} (x_k x_{k+1}) + (-1)^{c-1} (x_c w_0)\right\} \\ &+ (-1)^c \sum_{1 \leqslant j \leqslant 2b} (-1)^j (w_j w_{j+1}) \\ &- \left\{-(v_1 y_1) + \sum_{1 \leqslant i \leqslant d-1} (-1)^{\ell-1} (y_\ell y_{\ell+1}) + (-1)^{d-1} (y_d w_1)\right\}.\end{aligned}$$

Then one can easily check that $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbf{K}(G)$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. Note that both $\mathbf{u}_1$ and $\mathbf{u}_2$ are of height one.

Case 2: $c + d$ is odd: Let

$$\begin{aligned}\mathbf{u}'_1 = & \sum_{1 \leqslant i \leqslant 2a} (-1)^i (v_i v_{i+1}) \\ & + \left\{ -(v_0 x_1) + \sum_{1 \leqslant k \leqslant c-1} (-1)^{k-1} (x_k x_{k+1}) + (-1)^{c-1} (x_c w_0) \right\} \\ & + (-1)^c w_0 w + \left\{ (v_1 y_1) + \sum_{1 \leqslant l \leqslant d-1} (-1)^l (y_\ell y_{\ell+1}) + (-1)^d (y_d w_1) \right\}.\end{aligned}$$

$$\begin{aligned}\mathbf{u}'_2 = & (-1)^c \sum_{1 \leqslant j \leqslant 2b} (-1)^j (w_j w_{j+1}) \\ & + \left\{ -(v_0 x_1) + \sum_{1 \leqslant k \leqslant c-1} (-1)^{k-1} (x_k x_{k+1}) + (-1)^{c-1} (x_c w_0) \right\} \\ & + (v_0 v_1) - \left\{ (v_1 y_1) + \sum_{1 \leqslant \ell \leqslant d-1} (-1)^\ell (y_\ell y_{\ell+1}) + (-1)^d (y_d w_1) \right\}.\end{aligned}$$

Then one can also check that $\mathbf{u}'_1, \mathbf{u}'_2 \in \mathbf{K}(G)$ and $\mathbf{u} = \mathbf{u}'_1 + \mathbf{u}'_2$. Note that both $\mathbf{u}'_1$ and $\mathbf{u}'_2$ are of height one. Thus our proof of Theorem 3.4 is completed. □

Remark 3.5. As is mentioned in Introduction and in Remark 2.2, our problem has emerged through our study of Hodge cycles on certain abelian varieties of CM-type. In view of the fact that there is only one example of two-degenerate abelian variety of CM-type, which is constructed by White [7] and is of dimension 100, we are led to looking for two-degenerate abelian varieties of CM-type of smaller dimension with the help of graphs. The content of Theorem 3.4, however, tells us that we could find no two-degenerate abelian varieties as far as we search the world of graphs for them.

4. Two-dominatedness for graphs

In this section we show the two-dominatedness for two infinite families of graphs, namely the complete graph K_m with m vertices and the complete bipartite graph $K_{m,m}$ consisting of two classes with m elements.

In the course of proof of Theorem 3.4, we have expressed a certain element of height two in $\mathbf{K}(G)$ as the sum of two elements of height one. Since the weight of the latter elements are rather large, one might think that $w(G)$ tends to be large for every graph. We, however, prove a result which suggests that under a certain regularity condition for graphs the number could be unexpectedly small. More precisely we show the following.

Theorem 4.1. *The complete graph K_m and the complete bipartite graph $K_{m,m}$ are two-dominated for any $m \geqslant 3$.*

Remark 4.2. As will be noted in Remark 4.3, this phenomena should not be regarded as an isolated one. Apart from cyclic graphs with an even number $2n$ of vertices, which is clearly n-dominated, it is rather difficult to construct w-dominated graphs with $w \geqslant 3$.

Proof of Theorem 4.2. The proof for the complete graph K_m with m vertices is essentially contained in [5]. More precisely, noting that K_m is identified with the set of edges of the $(m-1)$-simplex in the real $(m-1)$-space $\mathbf{R}^{m-1}$, we infer the validity of our theorem from [5, Theorem 3.2]. Hence, we are left with the complete bipartite graph $K_{m,m}$. Recall the following lemma which relates the kernel of the incidence matrix of a graph with a certain eigenspace of the adjacency matrix of its line graph.

Lemma 4.2.1 (Cvetkovic et al. [3, 2.6.4])**.** *Let $G=(V,E)$ be a graph and $L(G)$ its line graph. Let $B(G)$ be the incidence matrix of G and $A(L(G))$ the adjacency matrix of $L(G)$. Then an element $\mathbf{e} \in \mathbf{E}(G)$ belongs to the kernel of $B(G)$ if and only if $\mathbf{e}$ is an eigenvector of $A(L(G))$ with eigenvalue -2.*

Therefore, we are reduced to the consideration of the eigenspace of $L_2(m) = L(K_{m,m})$ with eigenvalue -2. Note that we can identify the set of vertices of $L_2(m)$ with the set $\{(i,j); 1 \leqslant i,j \leqslant m\}$. It is acted upon by the product $S_m \times S_m$ of the mth symmetric group. Then as a representation space of $S_m \times S_m$, the $\mathbf{Q}$-vector space $\mathbf{V}_Q = \mathbf{V}(L_2(m)) \otimes \mathbf{Q}$ is isomorphic to the tensor product $\mathbf{Q}^m \otimes \mathbf{Q}^m$ of the natural representation $\mathbf{Q}^m$ on which S_m acts by the permutation of coordinates. Therefore, it is decomposed into irreducibles as

$$\begin{aligned}\mathbf{V}(L_2(m)) \otimes \mathbf{Q} &\cong ([m-11] \oplus [m]) \otimes ([m-11] \oplus [m]) \\ &\cong ([m-11] \oplus [m-11])) \oplus ([m-11] \oplus [m]) \\ &\quad \oplus([m] \oplus [m-11]) \oplus ([m] \oplus [m]),\end{aligned}$$

where $[a]$ denotes the representation corresponding to the Young diagram (a). Let us determine how the (-2)-eigenspace $\mathbf{V}_{-2}$ decomposes into irreducibles as a representation space of $S_m \times S_m$. Recall that the irreducible representation $[m-11]$ is realized as the $(m-1)$-dimensional subspace $\{(x_i) \in \mathbf{Q}^m; \sum_{1 \leqslant i \leqslant m} x_i = 0\}$, which has a basis $\{e_i - e_m; 1 \leqslant i \leqslant m-1\}$. Note that the element

$$\begin{aligned}&s(a,b) = (a,b) - (a,m) - (m,b) + (m,m) \in \mathbf{V}(L_2(m)) \ (= \mathbf{E}(K_{m,m})), \\ &1 \leqslant a,b \leqslant m-1,\end{aligned}$$

belongs to the kernel $\mathbf{K}(K_{m,m})$, and that through the natural isomorphism $\Phi : \mathbf{V}(L_2(m)) \otimes \mathbf{Q} \to \mathbf{Q}^m \otimes \mathbf{Q}^m$ defined by $\Phi(i,j) = e_i \otimes e_j$, this element corresponds to $(e_a - e_m) \otimes (e_b - e_m)$. Therefore, we see that the $\mathbf{Q}$-space $\langle s(a,b); 1 \leqslant a,b \leqslant m \rangle_Q$ spanned by these elements gives rise to the irreducible representation $[m-11]\otimes[m-11]$. Hence we obtain the following:

Lemma 4.2.2. *The kernel of the* $m^2\times m^2$-matrix $A(L_2(m))+2E_{m^2}$ *contains the subspace* $\langle s(a,b): 1\leqslant a,b\leqslant m\rangle_Q$, *which is isomorphic to the irreducible representation* $[m-11]\otimes[m-11]$ *of* $S_m\times S_m$ *of degree* $(m-1)^2$.

On the other hand, the row-space W of the $m^2\times m^2$-matrix $A(L_2(m))+2E_{m^2}$ is naturally identified with the vector space spanned by the m^2 square matrices $M(a,b)$, $1\leqslant a$, $b\leqslant m$, defined by

$$M(a,b)_{ij}=\begin{cases}2 & \text{when } (i,j)=(a,b),\\ 1 & \text{when } i=a \textit{ or } j=b, \text{ but } (i,j)\neq(a,b),\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.2.3. (i) *For any distinct pairs* $(a,b),(c,d)$ *with* $1\leqslant a,b,c,d\leqslant m$, *the relation*

$$(R_{(a,b;c,d)})\quad M(a,b)-M(a,d)-M(c,b)+M(c,d)=0$$

holds.

(ii) *The* $2m-1$ *matrices* $M(i,m)$ $(1\leqslant i\leqslant m-1)$, $M(m,j)$ $(1\leqslant j\leqslant m-1)$, *and* $M(m,m)$ *are linearly independent*.

Proof of Lemma 4.2.3. The validity of the relation $(R_{(a,b;c,d)})$ is easily checked. As for the assertion (ii), we proceed as follows. Let us assume that a linear relation

$$\begin{aligned}&\sum_{1\leqslant i\leqslant m-1} f(i,m)M(i,m)\\ &+\sum_{1\leqslant j\leqslant m-1} f(m,j)M(m,j)\\ &+f(m,m)M(m,m)=0 \qquad (4.1)\end{aligned}$$

holds for some rational numbers $f(i,m)$ $(1\leqslant i\leqslant m-1)$, $f(m,j)$ $(1\leqslant j\leqslant m-1)$, and $f(m,m)$. Considering the (i,j)th component of relation (4.1), we see that the equality

$$f(i,m)+f(m,j)=0$$

holds whenever $1\leqslant i,j\leqslant m-1$. This means that there exists an $r\in\mathbf{Q}$ such that $f(i,m)=r$, $1\leqslant i\leqslant m-1$. $f(m,j)=-r, 1\leqslant j\leqslant m-1$. Then the matrix on the left-hand side of (4.1) becomes

$$\begin{pmatrix} & & & rm\\ & O & & rm\\ & & & rm\\ -rm & -rm & -rm & 2f(m,m)\end{pmatrix}.$$

Hence (4.1) implies that $r=0$ and $f(m,m)=0$. Thus the proof of Lemma 4.2.3 is completed.

It follows from this lemma that the row–space W is of dimension $2m-1=m^2-(m-1)^2$. Hence, combining Lemmas 4.2.2 and 4.2.3, we arrive at the following:

Lemma 4.2.4. *As a representation space of* $S_m \times S_m$, *the row–space of* $A_{L_2(m)} + 2E_{m^2}$ *is decomposed as* $([m-11]\otimes[m])\oplus([m]\otimes[m-11])\oplus([m]\otimes[m])$, *and the kernel is isomorphic to* $[m-11]\otimes[m-11]$. *In particular*, *the kernel is spanned by the elements* $s(a,b)$, $1\leqslant a,b\leqslant m$.

Thus we have finished the proof of Theorem 4.2. □

Remark 4.3. The line graphs corresponding to the two families of graphs considered in the theorem are members of *strongly regular graphs with least eigenvalue* -2 (see [2]). Such graphs are completely classified as in (loc.cit., Theorem 4, 14). For the eigenspace with eigenvalue -2 can we formulate a similar problem. Actually we can show that all of such graphs, except O_3 and one of the *Chang graphs*, are two-dominated and the latter two graphs, which are *not* line graphs, are three-dominated.

References

[1] B. Bollobas, Modern Graph Theory, Graduate Tests in Mathematics, Vol. 184, Springer, Berlin, 1998.
[2] P.J. Cameron, J.H. van Lint, Designs, Graphs, Codes and their Links, London Mathematical Society Student Texts, Vol. 22, Cambridge University Press, Cambridge, 1991.
[3] D. Cvetkovic, P. Rowlinson, S. Simic, Eigenspaces of Graphs, Encyclopedia of Mathematics and its Application, Vol. 66, Cambridge University Press, Cambridge, 1997.
[4] R. Hartshorne, Algebraic Geometry, Graduate Tests in Mathematics, Vol. 52, Springer, Berlin, 1977.
[5] F. Hazama, On the kernels of the lowering operators for ranked posets, Far East J. Math. Sci. 3 (2001) 513–541.
[6] R.H. Villarreal, Rees algebras of edge ideals, Comm. Algebra 23 (1995) 3513–3524.
[7] S. P. White, Sporadic cycles on CM abelian varieties, Compositio Math. 88 (1993) 123–144.